Generalized susceptibilities for a perfect quantum gas

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Abstract: The system we consider here is a charged fermions gas in the effective mass approximation, and in grand-canonical conditions. We assume that the particles are confined in a three dimensional cubic box Λ with side $L \geq 1$, and subjected to a constant magnetic field of intensity $B \geq 0$. Define the grand canonical generalized susceptibilities χ_L^N, $N \geq 1$, as successive partial derivatives with respect to B of the grand canonical pressure P_L. Denote by P_∞ the thermodynamic limit of P_L. Our main result is that χ_L^N admit as thermodynamic limit the corresponding partial derivatives with respect to B of P_∞. In this paper we only give the main steps of the proofs, technical details will be given elsewhere.

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1 Introduction and results

In this paper, we are interested in the thermodynamic behavior of perfect Fermi gas in the presence of a constant magnetic field $\mathbf{B}$ at temperature $T > 0$ and chemical potential μ fixed. Although the particles have electric charge so that they can interact with the external magnetic field, we neglect all self-interactions and work in the effective mass approximation. We also neglect the spin, since it does not change the nature of our results.

Consider that the gas is confined in a three dimensional cubic box Λ of side $L \geq 1$, centered at the origin. The constant magnetic field is $\mathbf{B} := B\mathbf{e}_3$

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where $\mathbf{e}_3 := (0,0,1)$ is the third vector of the canonical base of $\mathbb{R}^3$. Denote with c the speed of light, e the electric charge which is supposed the same for each particle, and define the Larmor frequency $\omega := (\frac{e}{c})B \geq 0$. We associate to $\mathbf{B}$ the magnetic vector potential $\mathbf{a}$ defined by:

$$B\mathbf{a}(\mathbf{x}) := \frac{B}{2}\,\mathbf{e}_3 \wedge \mathbf{x}. \tag{1}$$

The operator $H(\omega) := \frac{1}{2}\left(-i\nabla - \omega\mathbf{a}\right)^2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ [K]. Denote by $H_L(\omega)$ the restriction of this operator to $C_0^\infty(\Lambda)$. The Hamiltonian of our one-particle-problem is the self-adjoint extension of $H_L(\omega)$ with Dirichlet boundary conditions. We will use the same notation for $H_L(\omega)$ and for its self-adjoint extension.

Let $z = e^{\beta\mu}$ be the fugacity (here $\beta = 1/T > 0$). We will allow z to take complex values, i.e. $z \in D := \mathbb{C}\setminus\,]-\infty, -e^{\frac{\beta\omega}{2}}]$. The grand canonical pressure P_L is then given from the grand canonical partition function Ξ_L by

$$P_L(\beta,z,\omega) = \frac{1}{\beta L^3}\ln \Xi_L(\beta,z,\omega) = \frac{1}{\beta L^3}\mathrm{Tr}\left[\ln\left(\mathbf{1} + z e^{-\beta H_L(\omega)}\right)\right]. \tag{2}$$

Define for $\omega > 0$,

$$P_\infty(\beta,z,\omega) = \omega(2\pi\beta)^{-\frac{3}{2}}\sum_{k=0}^{\infty} f_{\frac{3}{2}}(z e^{-(k+\frac{1}{2})\omega\beta})$$

and for $\omega = 0$,

$$P_\infty(\beta,z,0) = \beta^{-1}(2\pi\beta)^{-\frac{3}{2}}\, f_{\frac{5}{2}}(z)$$

where $f_\alpha(z)$ are the standard Fermi functions (see e.g. [A-C]). It is proved in [A-C] that $P_L(\beta,z,\omega)$ admits $P_\infty(\beta,z,\omega)$ as thermodynamic limit in the following sense: for all K compact included in D, and $\omega \geq 0$, one has

$$\lim_{L\to\infty}\sup_{z\in K}\left|P_L(\beta,z,\omega) - P_\infty(\beta,z,\omega)\right| = 0. \tag{3}$$

Yet it is known that if we define the grand canonical density $\rho_L(\beta,z,\omega) := \beta z\frac{\partial P_L}{\partial z}(\beta,z,\omega)$ and $\rho_\infty(\beta,z,\omega) := \beta z\frac{\partial P_\infty}{\partial z}(\beta,z,\omega)$, one also has under same assumptions

$$\lim_{L\to\infty}\sup_{z\in K}\left|\rho_L(\beta,z,\omega) - \rho_\infty(\beta,z,\omega)\right| = 0. \tag{4}$$

The grand canonical generalized susceptibilities of a gas of fermions are defined by:

$$\chi_L^N(\beta,z,\omega) := \frac{\partial^N P_L}{\partial\omega^N}(\beta,z,\omega), \quad N \geq 1. \tag{5}$$

Notice that $\chi_L^1(\beta,z,\omega)$ is the magnetization of the system, and $\chi_L^2(\beta,z,\omega)$ is the magnetic susceptibility. In case $\omega=0$ and $L\to\infty$, it is known in the physical literature (see [A-B-N 1] for the rigorous proof) that $\chi_\infty^1(\beta,z,0)=0$ and $\chi_\infty^2(\beta,z,0)\neq 0$. Thus at zero field, the magnetic response is quadratic. In case $\omega\neq 0$, the magnetization is not zero, thus the magnetic response becomes linear. However, the rigorous treatment of this question needs good estimates on the higher correction terms (χ_L^N, $N\geq 2$) in the $\omega-$ expension of the pressure.

Our main result is that the generalized susceptibilities admit a thermodynamic limit in the following sense:

Theorem 1.1. *Fix $N\in\mathbb{N}^*$, $\beta>0$ and $\omega\geq 0$. Then for every compact K included in $D=\mathbb{C}\backslash\,]-\infty,-e^{\frac{\beta\omega}{2}}]$, one has*

$$\lim_{L\to\infty}\sup_{z\in K}\left|\chi_L^N(\beta,z,\omega)-\chi_\infty^N(\beta,z,\omega)\right|=0. \tag{6}$$

A straightforward consequence of this result is that under the same assumptions:

$$\forall N\in\mathbb{N}^*,\ \forall m\in\mathbb{N},\ \lim_{L\to\infty}\sup_{z\in K}\left|\frac{\partial^m\chi_L^N}{\partial z^m}(\beta,z,\omega)-\frac{\partial^m\chi_\infty^N}{\partial z^m}(\beta,z,\omega)\right|=0. \tag{7}$$

Having uniform limits with respect to z is very useful if one wants to translate this type of results in the canonical ensemble. See for example [C 1] and [C 2] for further ideas in this direction.

Now let us mention some previous works dealing with similar problems. One of truly rigorous results for the case $\omega=0$ and $N=1,2$ was given by Angelescu *et al* in [A-B-N 1]. Then Macris *et al* in [M-M-P] discussed the case when ω was arbitrary but $N=1$ and $|z|<1$. In [C 1] this condition on z was lifted for the case of Bose statistics (but this result can be immediately translated for the Fermi case). Let us further remark that we can adapt our techniques to the case when Λ is more general than just a box. We can in fact allow any domain with regular enough boundary which converges to $\mathbb{R}^3$ in the sense of Fischer. Moreover, our results hold for all ω. In comparison, the method used in [A-B-N 1] was strongly conditioned by the parallelipipedic geometry of Λ and by $\omega=0$.

We know that the physical importance of χ_L^N for $N\geq 3$ becomes relevant only when the deviations are large. But the mathematical proof of their convergence is not much more complicated than in the case when $N=2$, and it shows that there is no accident that one had convergence when $N=1$ or $N=2$. Concluding, our present result settles the question for derivatives of all order, for all Larmor frequencies and for all fugacities.

We would like to remind the reader that this paper only contains very basic ideas about proofs, and is mainly intended to give a detailed overview about the long and rather complicated technical steps that are needed.

2 Strategy

Let us recall Vitali-Porter theorem [H-P]:

Theorem 2.1. *Let $\{f_L\}_{L\geq 1}$ be a family of holomorphic functions on a fixed domain $D \subseteq \mathbb{C}$. Assume that $|f_L(z)| \leq M$ for all $L \geq 1$ and all $z \in D$. Assume also the existence of a subset $D' \subseteq D$ having an accumulation point $z_0 \in D$, such that $\lim_{L\to\infty} f_L(z)$ exists for each $z \in D'$. Then $\lim_{L\to\infty} f_L(z)$ exists everywhere in D, the convergence is uniform with respect to z in any compact subset of D and the limit function $f_\infty(z)$ is holomorphic in D.*

In our case, we have $D = \mathbb{C}\backslash\,]-\infty, -e^{\frac{\beta\omega}{2}}]$, $D' = \{z \in \mathbb{C} : |z| < 1\}$, $f_L(\cdot) = \chi_L^N(\beta, \cdot, \omega)$, and $f_\infty(\cdot) = \chi_\infty^N(\beta, \cdot, \omega)$.

Therefore, to prove the Theorem 1.1, we need to show first the pointwise convergence on the unit open disk

$$\lim_{L\to\infty} \chi_L^N(\beta, z, \omega) = \chi_\infty^N(\beta, z, \omega), \quad |z| < 1, \tag{8}$$

and second, that for every compact $K \subset D$ one has the uniform bound in $L \geq 1$

$$\sup_{z\in K} \left|\chi_L^N(\beta, z, \omega)\right| \leq \text{const}(\beta, K, \omega). \tag{9}$$

Then Theorem 1.1 would be proven.

3 Elements of proofs

3.1 The pointwise limit: proof of (8)

Denote by $(\mathcal{I}_1(L^2(\Lambda)), \|\cdot\|_{\mathcal{I}_1})$ the Banach space of trace class operators. It is well known that for any $\omega \in \mathbb{R}$, the family of operators $\{W_L(\beta, \omega)\}_{\beta>0} = \{e^{-\beta H_L(\omega)}\}_{\beta>0}$ is a Gibbs semigroup [H-P] (the operators $H_L(\omega)$ are self-adjoint, positive and $\{W_L(\beta, \omega)\}_{\beta>0} \subset \mathcal{I}_1(L^2(\Lambda))$). On the other hand W_L has an integral kernel $G_L(\mathbf{x}, \mathbf{x}', \beta, \omega)$ which is continuous on $\Lambda \times \Lambda$ with respect to spatial variables $\mathbf{x}$ and $\mathbf{x}'$. The diamagnetic inequality reads as:

$$|G_L(\mathbf{x}, \mathbf{x}', \beta, \omega)| \leq \frac{1}{(2\pi\beta)^{3/2}}\, e^{-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\beta}}, \quad \mathbf{x}, \mathbf{x}' \in \Lambda, \tag{10}$$

which then implies that

$$\|W_L\|_{\mathcal{I}_1} = \operatorname{Tr} W_L = \int_\Lambda G_L(\mathbf{x},\mathbf{x},\beta,\omega)d\mathbf{x} \leq \frac{L^3}{(2\pi\beta)^{\frac{3}{2}}} \tag{11}$$

Suppose z in the unit disk D', from (2) we have (see e.g. (2.10) and (2.11) in [C 1] in the case of Fermi statistics):

$$P_L(\beta,z,\omega) = \frac{1}{\beta L^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^n}{n}\operatorname{Tr}\left(W_L\left(n\beta,\omega\right)\right). \tag{12}$$

We denote by $G_\infty(\mathbf{x},\mathbf{x}',\beta,\omega)$ the integral kernel of the corresponding operator defined on the whole space: $\{e^{-\beta H(\omega)}\}_{\beta>0}$ (see (2.2) in [A-C] or (4.90) in [C 1]). Its diagonal is very simple and is given by

$$G_\infty(\mathbf{x},\mathbf{x},\beta,\omega) = \frac{1}{(2\pi\beta)^{3/2}}\frac{\omega\beta/2}{\sinh(\omega\beta/2)}.$$

We remark that this quantity is $\mathbf{x}-$independent and in view of (2) we can write

$$P_\infty(\beta,z,\omega) = \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^n}{n}\;G_\infty(\mathbf{x},\mathbf{x},n\beta,\omega). \tag{13}$$

We are interested in derivatives of P_L with respect to ω. Due to formula (12), these derivatives will act on the trace of the semigroup. We are thus motivated to study the $\mathcal{I}_1$-analyticity with respect to ω of the semigroup. Although this result was already proven in [A-B-N 1], we state it in Lemma 3.1, since we will use it later on.

In order to do that, we need to introduce further notation. Define the following operators by their corresponding integral kernels:

$$\begin{aligned}
\hat{R}_{1,L}(\mathbf{x},\mathbf{x}',\beta,\omega) &:= \mathbf{a}(\mathbf{x})\cdot(i\nabla_{\mathbf{x}}+\omega\mathbf{a}(\mathbf{x}))G_L(\mathbf{x},\mathbf{x}',\beta,\omega),\\
\hat{R}_{2,L}(\mathbf{x},\mathbf{x}',\beta,\omega) &:= \frac{1}{2}\,\mathbf{a}^2(\mathbf{x})\,G_L(\mathbf{x},\mathbf{x}',\beta,\omega).
\end{aligned} \tag{14}$$

The operators $\hat{R}_{1,L}(\beta,\omega)$ and $\hat{R}_{2,L}(\beta,\omega)$ are of trace class as well as the operators,

$$I_{n,L}(i_1,...,i_n)(\beta,\omega) := \int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2...\int_0^{\tau_{n-1}}d\tau_n\,W_L(\beta-\tau_1,\omega) \tag{15}$$
$$\cdot\hat{R}_{i_1,L}(\tau_1-\tau_2,\omega)\hat{R}_{i_2,L}(\tau_2-\tau_3,\omega)...\hat{R}_{i_{n-1},L}(\tau_{n-1}-\tau_n,\omega)\hat{R}_{i_n,L}(\tau_n,\omega)$$

for $n \geq 1$ and for $(i_1, ..., i_n) \in \{1, 2\}^n$. We can finally give the analyticity result, define

$$c_n^N(i_1, ..., i_n) := \begin{cases} 1 & \text{if } i_1 + ... + i_n = N \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Lemma 3.1. *The operator-valued function $\mathbb{R} \ni \omega \mapsto W_L(\beta, \omega) \in \mathcal{I}_1$ admits an entire extension to $\mathbb{C}$. Fix $\omega_0 \in \mathbb{R}$. For all $\omega \in \mathbb{C}$ we have*

$$\begin{aligned} W_L(\beta, \omega) &= \sum_{N=0}^{\infty} \frac{(\omega - \omega_0)^N}{N!} \frac{\partial^N W_L}{\partial \omega^N}(\beta, \omega_0), \qquad (17) \\ \frac{\partial^N W_L}{\partial \omega^N}(\beta, \omega_0) &= N! \sum_{n=1}^{N} (-1)^n \sum_{i_j \in \{1,2\}} c_n^N(i_1, ..., i_n) I_{n,L}(i_1, ..., i_n)(\beta, \omega_0). \end{aligned}$$

This implies in particular that the traces of the semigroup W_L which appear in formula (12) are entire functions of ω.

Remark 3.2. *It is important to notice that the expansion (17) is not really convenient if one wants to prove (8). That is because the expressions (14) contain at least one term as $\mathbf{a}(\mathbf{x})$ which behaves like $\mathbf{x}$. So direct estimates show that the trace norm of $\frac{\partial^N W_L}{\partial \omega^N}(\beta, \omega_0)$ (see e.g. [A-B-N 2]) behaves like L^{3+N}, and this is very far from the desired behavior of L^3. It is true that when we look at the trace and not at the trace-norm, things are quite different. In [A-B-N 1] it is proved at $\omega_0 = 0$ and for $N = 1, 2$ that due to some remarkable identities, the terms growing like L^{3+N} are identically zero. What we do next in our paper is to give an alternative expansion which takes care of these singularities for all terms at the same time.*

In order to do that, we concentrate on the kernel G_L. We remark first the following

Lemma 3.3. *For every $\omega \in \mathbb{C}$, the operator $W_L(\beta, \omega)$ defined by the series in (17) admits an integral kernel $G_L(\mathbf{x}, \mathbf{x}', \beta, \omega)$. This kernel is defined as the sum of a series as in (17) where instead of operators we consider their integral kernels. Then G_L is continuous with respect to the spatial variables, and is an entire function of ω. In addition, for all $\mathbf{x}, \mathbf{x}' \in \Lambda$ fixed, one has*

$$\frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x}, \mathbf{x}', \beta, \omega) = \left(\frac{\partial^N W_L}{\partial \omega^N} \right)(\mathbf{x}, \mathbf{x}', \beta, \omega). \tag{18}$$

Hints to the proof. Notice that $\frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x}, \mathbf{x}')$ is kernel's derivative, while $\frac{\partial^N W_L}{\partial \omega^N}(\mathbf{x}, \mathbf{x}')$ denotes the kernel of the trace class operator $\frac{\partial^N W_L}{\partial \omega^N}$. The idea of the proof consists in showing that when replacing $I_{n,L}$ from (17) with its integral kernel (defined as a continuous function in $\mathbf{x}, \mathbf{x}' \in \overline{\Lambda}$ by the multiple convolution (15)), the power series in (17) converges uniformly for $\mathbf{x}, \mathbf{x}' \in \overline{\Lambda}$, and has an infinite radius of convergence. The estimates rely on the diamagnetic inequality (10) and an induction argument. $\square$

Since $\frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x}, \mathbf{x}', \beta, \omega)$ is continuous for $(\mathbf{x}, \mathbf{x}') \in \Lambda \times \Lambda$, and $\frac{\partial^N W_L}{\partial \omega^N}$ is a trace class operator, its trace can be expressed as the integral of the diagonal of its kernel (see the remark at page 523 in [K]). We conclude that for every $\omega \in \mathbb{C}$:

$$\frac{\partial^N}{\partial \omega^N} \mathrm{Tr}\left(W_L\left(\beta, \omega\right)\right) = \mathrm{Tr}\left(\frac{\partial^N W_L}{\partial \omega^N}\left(\beta, \omega\right)\right) = \int_\Lambda \frac{\partial^N G_L}{\partial \omega^N}\left(\mathbf{x}, \mathbf{x}, \beta, \omega\right) d\mathbf{x}. \quad (19)$$

In the light of Remark 3.2, we need a different formula for the above kernel, so that the apparent growing terms cancel each other. This will be done by using a modified perturbation theory for magnetic Gibbs semigroups. Previous works which dealt with similar problems are [C-N], [C 1], [B-C] and [N].

For that, we introduce the magnetic phase ϕ and the magnetic flux fl where for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Lambda$, and as before $\mathbf{e}_3 = (0, 0, 1)$,

$$\phi(\mathbf{x}, \mathbf{y}) := \frac{1}{2}\mathbf{e}_3 \cdot (\mathbf{y} \wedge \mathbf{x}), \quad \mathrm{fl}(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \phi(\mathbf{x}, \mathbf{y}) + \phi(\mathbf{y}, \mathbf{z}) + \phi(\mathbf{z}, \mathbf{x}). \quad (20)$$

We have

$$|\mathrm{fl}(\mathbf{x}, \mathbf{y}, \mathbf{z})| \leq |\mathbf{x} - \mathbf{y}|\, |\mathbf{y} - \mathbf{z}|. \quad (21)$$

For every $n \geq 1$ and for every points $\mathbf{x}, \mathbf{y}_1, ..., \mathbf{y}_n \in \Lambda$, we introduce

$$\mathrm{Fl}_1(\mathbf{x}, \mathbf{y}_1) = 0, \quad \mathrm{Fl}_n(\mathbf{x}, \mathbf{y}_1, ..., \mathbf{y}_n) = \sum_{k=1}^{n-1} \mathrm{fl}(\mathbf{x}, \mathbf{y}_k, \mathbf{y}_{k+1}),\ n \geq 2.$$

Fix $\omega > 0$. Consider now the bounded operators given by their integral kernels:

$$\begin{aligned} R_{1,L}(\mathbf{x}, \mathbf{x}', \beta, \omega) &:= \mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot (i\nabla_\mathbf{x} + \omega \mathbf{a}\left(\mathbf{x}\right))\, G_L(\mathbf{x}, \mathbf{x}', \beta, \omega), \\ R_{2,L}(\mathbf{x}, \mathbf{x}', \beta, \omega) &:= \frac{1}{2}\mathbf{a}^2(\mathbf{x} - \mathbf{x}') G_L(\mathbf{x}, \mathbf{x}', \beta, \omega) \end{aligned} \quad (22)$$

and for all $\mathbf{x} \in \Lambda$, $n \geq 1$ and $k \geq 0$

$$W_{L,n}^{k}(\mathbf{x},\mathbf{x},\beta,\omega) := \sum_{j=1}^{n}(-1)^j \sum_{(i_1,...,i_j)\in\{1,2\}^j} c_j^n(i_1,...,i_j)\int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots$$
$$\int_0^{\tau_{j-1}} d\tau_j \int_\Lambda d\mathbf{y}_1 \ldots \int_\Lambda d\mathbf{y}_j \, \frac{\left(i\left(\mathrm{Fl}_j(\mathbf{x},\mathbf{y}_1,...,\mathbf{y}_j)\right)\right)^k}{k!} \, G_L(\mathbf{x},\mathbf{y}_1,\beta-\tau_1,\omega)$$
$$R_{i_1,L}(\mathbf{y}_1,\mathbf{y}_2,\tau_1-\tau_2,\omega) \ldots R_{i_{j-1},L}(\mathbf{y}_{j-1},\mathbf{y}_j,\tau_{j-1}-\tau_j,\omega) R_{i_j,L}(\mathbf{y}_j,\mathbf{x},\tau_j,\omega). \quad (23)$$

By convention, in the case when $k=0$ we set $0^0 \equiv 1$.

The next lemma gives a new expression for the diagonal of kernel's N-th derivative with respect to ω at finite volume.

Lemma 3.4. *Fix $\omega_0 \geq 0$. Then for all $\mathbf{x} \in \Lambda$, and for all $N \in \mathbb{N}^*$, one has*

$$\frac{1}{N!}\frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x},\mathbf{x},\beta,\omega_0) = \sum_{n=1}^{N} W_{L,n}^{N-n}(\mathbf{x},\mathbf{x},\beta,\omega_0). \quad (24)$$

Hints to the proof. The proof heavily relies on the general theory developed in [A-B-N 2] and in [C 1], where a version of Duhamel's formula is written for the perturbed semigroup $W_L(\beta,\omega)$ for $d\omega := \omega - \omega_0$ small, here $\omega_0 \geq 0$ is fixed (see Proposition 3 and formula (4.61) in [C 1]). Roughly speaking, one has to iterate this formula N times, and identify the term containing $(d\omega)^N$. □

Based on the above formula, we can give an expression for the corresponding quantities at infinite volume:

Lemma 3.5. *Fix $\omega_0 \geq 0$. Then for all $x \in \Lambda$, and for all $N \in \mathbb{N}^*$, one has*

$$\frac{1}{N!}\frac{\partial^N G_\infty}{\partial \omega^N}(\mathbf{x},\mathbf{x},\beta,\omega_0) = \lim_{L\to\infty} \frac{1}{N!}\frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x},\mathbf{x},\beta,\omega_0)$$
$$= \sum_{n=1}^{N}\sum_{j=1}^{n}(-1)^j \sum_{i_k\in\{1,2\}} c_j^n(i_1,...,i_j)\int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{j-1}} d\tau_j \int_{\mathbb{R}^3} d\mathbf{y}_1 \ldots$$
$$\int_{\mathbb{R}^3} d\mathbf{y}_j \, \frac{(i\mathrm{Fl}_j(\mathbf{x},\mathbf{y}_1,...,\mathbf{y}_j))^{N-n}}{(N-n)!} \, G_\infty(\mathbf{x},\mathbf{y}_1,\beta-\tau_1,\omega_0) R_{i_1,\infty}(\mathbf{y}_1,\mathbf{y}_2,\tau_1-\tau_2,\omega_0)$$
$$\ldots R_{i_{j-1},\infty}(\mathbf{y}_{j-1},\mathbf{y}_j,\tau_{j-1}-\tau_j,\omega_0) R_{i_j,\infty}(\mathbf{y}_j,\mathbf{x},\tau_j,\omega_0) \quad (25)$$

where $R_{j,\infty}$ are defined as in (22) with $L \equiv \infty$.

Hints to the proof. We need to estimate $(G_L - G_\infty)(\mathbf{x},\mathbf{x}')$ and $(i\nabla_\mathbf{x} + \omega_0\mathbf{a}(\mathbf{x}))(G_L - G_\infty)(\mathbf{x},\mathbf{x}')$. We have to take into account the walls' influence on the integral kernel at finite volume. We use a variant of Green's formula for the solutions of the heat equation inside Λ. Thus, we get the next result. Define for all $\mathbf{x} \in \Lambda$, $\delta_{\mathbf{x}} := \mathrm{dist}(\mathbf{x}, \partial\Lambda)$ and $M := \{\mathbf{x} \in \Lambda \;:\; \delta_{\mathbf{x}} \leq 1\}$. Then one has

$$\begin{aligned} &|(G_L - G_\infty)(\mathbf{x}, \mathbf{x}', \beta, \omega_0)|,\ |(i\nabla_{\mathbf{x}} + \omega_0 \mathbf{a}(\mathbf{x}))(G_L - G_\infty)(\mathbf{x}, \mathbf{x}', \beta, \omega_0)| \\ &\leq \frac{\text{const}}{\beta^2} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{c\beta}\right)\left(\chi_M(\mathbf{x}) + \chi_M(\mathbf{x}') + \exp\left(-\frac{\delta_{\mathbf{x}}^2}{c\beta} - \frac{\delta_{\mathbf{x}'}^2}{c\beta}\right)\right), \end{aligned} \tag{26}$$

where $c > 0$ is a constant and χ_M is the characteristic function of M. Then (25) follows after some straightforward calculations from (10), (26), (20), the estimate (4.46) in [C 1], and (24). □

Since our aim is to prove (8), we write from (12) and (13)

$$(P_L - P_\infty)(\beta, z, \omega) = \frac{1}{\beta L^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n} \int_\Lambda d\mathbf{x}\, (G_L - G_\infty)(\mathbf{x}, \mathbf{x}, n\beta, \omega). \tag{27}$$

In order to conclude that (8) is true, it will be sufficient to show that

Lemma 3.6. *Fix $\omega_0 \geq 0$. Then for all $N \in \mathbb{N}^*$, one has*

$$\left| \int_\Lambda d\mathbf{x} \left(\frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x}, \mathbf{x}, \beta, \omega_0) - \frac{\partial^N G_\infty}{\partial \omega^N}(\mathbf{x}, \mathbf{x}, \beta, \omega_0) \right) \right| \leq L^2 f(\beta, \omega_0, N), \tag{28}$$

where $f(\cdot, \omega_0, N)$ is a function of β which is polynomially bounded.

Hints to the proof. Fix $N \in \mathbb{N}^*$. One denotes by $F_{L,N}(\mathbf{x}, \beta, \omega_0)$ the formula obtained by replacing in formula (25) all the spatial integrals on $\mathbb{R}^3$ by integrals on Λ. To estimate the difference $F_{L,N}(\mathbf{x}, \beta, \omega_0) - \frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x}, \mathbf{x}, \beta, \omega_0)$, we use (24), (10), (26), (20), and the estimate (4.46) in [C 1]. Finally we find that

$$\left| \int_\Lambda d\mathbf{x} \left(F_{L,N}(\mathbf{x}, \beta, \omega_0) - \frac{\partial^N G_L}{\partial \omega^N}(\mathbf{x}, \mathbf{x}, \beta, \omega_0) \right) \right| \leq L^2 f_1(\beta, \omega_0, N), \tag{29}$$

where f_1 is polynomially bounded with respect to β. Now we need to estimate the difference $F_{L,N}(\mathbf{x}, \beta, \omega_0) - \frac{\partial^N G_\infty}{\partial \omega^N}(\mathbf{x}, \mathbf{x}, \beta, \omega_0)$. From the definition of $F_{L,N}$, this difference will consist with integrals as in (25) where in at least one of the spatial integrals one integrates over $\mathbb{R}^3 \setminus \Lambda$. Since it can also be shown that (29) holds again if we replace G_L with G_∞, the lemma is proven up to the use of the triangle inequality which yields (28). □

Since we have seen in (19) that the derivatives with respect to ω and the trace commute at finite volume, formula (27) and Lemma 3.6 show that for every $|z| < 1$, the derivatives with respect to ω of $(P_L - P_\infty)(\beta, z, \omega)$ behave like $\frac{1}{L}$, which finishes the proof of (8).

3.2 The uniform bound: proof of (9)

If we denote by $g_L(\beta,\tau,\omega) = g_L(\beta,\tau,\omega,z,\xi) = (\xi - zW_L(\beta,\omega))^{-1} zW_L(\tau,\omega)$ then for $\beta > 0$, $z \in K \subset D$, $\omega \geq 0$, one has (see (4.2) in [C 1] for the Bose case):

$$P_L(\beta,z,\omega) = \frac{1}{2i\pi}\int_C d\xi\ \frac{\ln(1+\xi)}{\xi}\,\frac{1}{\beta L^3}\,\mathrm{Tr}\left(g_L\left(\beta,\beta,\omega\right)\right) \tag{30}$$

with $C \subset \mathbb{C}\backslash]-\infty,-1]$ surrounds the eigenvalues of zW_L. In addition, one chooses C in order to have $(\xi - zW_L(\beta,\omega))^{-1}$ bounded for every $\xi \in C$ and $z \in K$. We have seen that $\omega \mapsto W_L(\beta,\omega)$ is $\mathcal{I}_1$-analytic on $\mathbb{C}$; one can also see that $\mathrm{Tr}(g_L)$ is a real-analytic function of ω. In the end, the generalized susceptibilities are well defined as functions of z on D. We see that (9) will follow from

$$\sup_{z\in K}\sup_{\xi\in\mathcal{C}}\left|\frac{\partial^N}{\partial\omega^N}\,\mathrm{Tr}\left(g_L(\beta,\beta,\omega)\right)\right| \leq\ L^3\,\mathrm{const}(\beta,K). \tag{31}$$

Here we have a similar problem as the one pointed out for the semigroup in Remark 3.2. We could try to use the inequality

$$\left|\frac{\partial^N}{\partial\omega^N}\,\mathrm{Tr}\left(g_L(\beta,\beta,\omega)\right)\right| \leq \left\|\frac{\partial^N g_L}{\partial\omega^N}(\beta,\beta,\omega)\right\|_{\mathcal{I}_1},$$

but the right hand side behaves like L^{3+N} and not like L^3 as desired. What we do instead is finding a Taylor expansion *directly for the trace*, and to give the right estimate for its derivatives.

In view of developing $\mathrm{Tr}(g_L)$ as a function of ω in a small real neighborhood Ω of $\omega_0 \geq 0$, we first analyze g_L in $\mathcal{I}_1(L^2(\Lambda))$. As a general rule, for an integral operator $T(\omega_0)$ with kernel $t(\mathbf{x},\mathbf{x}',\omega_0)$, we denote by $\tilde{T}(\omega)$ the operator which has an integral kernel given by (see also (20))

$$\tilde{t}(\mathbf{x},\mathbf{x}',\omega) := e^{i(\omega-\omega_0)\phi(\mathbf{x},\mathbf{x}')}t(\mathbf{x},\mathbf{x}',\omega_0).$$

Lemma 3.7. *Fix $\omega_0 \geq 0$ and $N \geq 1$. Then for every $d\omega := \omega - \omega_0$ small, there exist N trace class operators $a_{L,n}(\beta,\omega) = a_{L,n}(\beta,\omega,z,\xi)$, $1 \leq n \leq N$, and an operator $R_{L,N+1}(\beta,\omega) = R_{L,N+1}(\beta,\omega,z,\xi)$ such that*

$$\begin{aligned} g_L(\beta,\beta,\omega) \ &= \ \widetilde{g_L}\left(\beta,\frac{\beta}{2},\omega\right)\widetilde{W_L}\left(\frac{\beta}{2},\omega\right) + \sum_{n=1}^{N} d\omega^n\, a_{L,n}(\beta,\omega) \\ &+ \ R_{L,N+1}(\beta,\omega), \end{aligned} \tag{32}$$

where

$$\begin{aligned}\|a_{L,n}(\beta,\omega)\|_{\mathcal{I}_1} &\leq \operatorname{const}(\beta,K,C)L^3, \quad \omega\in\Omega, \quad 1\leq n\leq N\\ \|R_{L,N+1}(\beta,\omega)\|_{\mathcal{I}_1} &\leq |d\omega|^{N+1}\operatorname{const}(\beta,K,C)L^3. \end{aligned} \tag{33}$$

Notice that the operators $a_{L,n}$ still depend on ω.

Hints to the proof. We use a technique which generalizes the one developed in [C 1], which only worked for $N=1$. Our generalization is considerably more involved than the original argument given in [C 1], which at its turn was rather lengthy. See for comparison formula (4.84) in [C 1], which corresponds to the case $N=1$ in our lemma. Full proofs will be given elsewhere. □

As (32) is valid in $\mathcal{I}_1(L^2(\Lambda))$, we can take the trace term by term in this equality. This gives (see also (4.85) in [C 1]):

$$\begin{aligned}\operatorname{Tr}(g_L(\beta,\beta,\omega)) &= \operatorname{Tr}(g_L(\beta,\beta,\omega_0))\\ &+ \sum_{n=1}^{N} d\omega^n\left(\operatorname{Tr}(a_{L,n}(\beta,\omega))\right)+\operatorname{Tr}\left(R_{L,N+1}(\beta,\omega)\right). \end{aligned} \tag{34}$$

The last technical result we need is contained in the following lemma, given again without proof:

Lemma 3.8. *Fix $\omega_0\geq 0$, $N\geq 1$ and $1\leq n\leq N$. Then there exists a family of ω-independent coefficients $\{b^m_{L,n}(\beta,\omega_0)\}_{m\in\mathbb{N}}=\{b^m_{L,n}(\beta,\omega_0,z,\xi)\}_{m\in\mathbb{N}}$, and a remainder $r^{N+1}_{L,n}(\beta,\omega)=r^{N+1}_{L,n}(\beta,\omega,z,\xi)$ such that for $d\omega=\omega-\omega_0$ small, one has*

$$\begin{aligned}\operatorname{Tr}\left(a_{L,n}(\beta,\omega)\right) &= \sum_{m=0}^{N} d\omega^m\, b^m_{L,n}(\beta,\omega_0)+r^{N+1}_{L,n}(\beta,\omega), \qquad (35)\\ |r^{N+1}_{L,n}(\beta,\omega)| &\leq L^3|d\omega|^{N+1}\operatorname{const}(\beta,K,C),\\ \left|b^m_{L,n}(\beta,\omega_0)\right| &\leq L^3\operatorname{const}(\beta,K,C). \end{aligned}$$

Consequently, from (34) and (35) we have

$$\left[\frac{\partial^N}{\partial\omega^N}\operatorname{Tr}(g_L)\right](\beta,\beta,\omega_0,z,\xi)=\sum_{n=1}^{N} b^{N-n}_{L,n}(\beta,\omega_0,z,\xi), \tag{36}$$

and (31) follows.

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